## Problem Based on Condition for Common Roots

**SHORT ANSWER TYPE QUESTIONS :**

**SA.1** If the equations $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ have a common root, then either $b = c$ or $b + c + 1 = 0$.

**Sol.** Let $\alpha$ be the common root of the equations

$\alpha^2 + b\alpha + c = 0$ .......(1)

and $\alpha^2 + c\alpha + b = 0$ .......(2)

Solving (1) and (2), we get

$$\frac{\alpha^2}{b^2 - c^2} = \frac{\alpha}{c - b} = \frac{1}{c - b}$$

Eliminating $\alpha$,

$(c - b)^2 = (b^2 - c^2)(c - b)$

$\Rightarrow$ $(c - b)^2 + (c^2 - b^2)(c - b) = 0$

$\Rightarrow$ $(c - b)[(c - b) + (c^2 - b^2)] = 0$

either $c - b = 0$, i.e. $b = c$

or $(c - b) - (b - c)(b + c) = 0$,

i.e. $(c - b)[1 + (b + c)] = 0$

either $b = c$ or $b + c + 1 = 0$.

**SA.2** Find the value of k if the equations $x^2 + 2x + 3k = 0$ and $2x^2 + 3x + 5k = 0$ have a common roots.

**Sol.** The given equations are

$x^2 + 2x + 3k = 0$ ........(1)

and $2x^2 + 3x + 5k = 0$ ........(2)

Let $\alpha$ be the common root, then it must satisfy both the equations. Therefore,

$\alpha^2 + 2\alpha + 3k = 0$ ........(3)

and $2\alpha^2 + 3\alpha + 5k = 0$ ........(4)

Hence, by cross-multiplication, we have

$$\frac{\alpha^2}{10k - 9k} = \frac{\alpha}{6k - 5k} = \frac{1}{3 - 4} \quad ........(5)$$

From the first-two members of (5),

$$\alpha = \frac{k}{k} = 1$$

From the last-two members of (5),

$$\alpha = \frac{k}{-1} = -k.$$

Equating the two values of $\alpha$,

$-1 = k$ or $k = -1$.

## Problem Based on Condition for Common Roots

**SA.3** If the equation $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root, prove that either $\frac{ps - qr}{q - s}$ or $\frac{q - s}{r - p}$.

**Sol.** The given equations are

$$x^2 + px + q = 0 \quad .........(1)$$

and $$x^2 + rx + s = 0 \quad .........(2)$$

Let $\alpha$ be the common root. Then

$$\alpha^2 + p\alpha + q = 0 \quad .........(3)$$

and $$\alpha^2 + r\alpha + s = 0 \quad .........(4)$$

By cross-multiplication,

$$\frac{\alpha^2}{ps - qr} = \frac{\alpha}{q - s} = \frac{1}{r - p} \quad .........(5)$$

From the first-two member; $\alpha = \frac{ps - qr}{q - s}$.

From the last-two member; $\alpha = \frac{q - s}{r - p}$.

Hence the common root is either $\frac{ps - qr}{q - s}$ or $\frac{q - s}{r - p}$.

**SA.4** If the equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root then prove that $a + b + c = 0$ or $a = b = c$, where a, b and c are reals.

**Sol.** Let $\alpha$ be the common root of the given equations then $\alpha$ will satisfy both the equations :

$$a\alpha^2 + b\alpha + c = 0 \text{ and } b\alpha^2 + c\alpha + a = 0 .$$

By the method of cross-multiplication,

$$\frac{\alpha^2}{ab - c^2} = \frac{\alpha}{bc - a^2} = \frac{1}{ac - b^2}.$$

or $$\left(\frac{bc - a^2}{ac - b^2}\right)^2 = \frac{ab - c^2}{ac - b^2}$$

or $(bc - a^2)^2 = (ab - c^2)(ac - b^2)$

or $b^2c^2 - 2a^2bc + a^4 = a^2bc - ab^3 - ac^3 + b^2c^2$

or $a^4 + ab^3 + ac^3 - 3a^2bc = 0$

or $a(a^3 + b^3 + c^3 - 3abc) = 0$

But $a \neq 0$;

$\therefore$ $a^3 + b^3 + c^3 - 3abc = 0$

or $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\therefore$ $a + b + c = 0$

or $a^2 + b^2 + c^2 - ab - bc - ca = 0$

Now $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow$ $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow$ $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

This is possible only when $a - b = 0$, $b - c = 0$, $c - a = 0$ hold simultaneously. That is, $a = b = c$.

## Problem Based on Condition for Common Roots

**SA.5** If the equation $x^2 - ax + b = 0$ and $x^2 - a_1x + b_1 = 0$, have one root in common and the second equation has equal roots, prove that $2(b + b_1) = aa_1$.

**Sol.** Since the equation $x^2 - a_1x + b_1 = 0$ has equal roots, let the roots be $\alpha$, $\alpha$. Then

$$\alpha + \alpha = 2\alpha = a_1$$

$$\Rightarrow \quad \alpha = \frac{a_1}{2} \text{ and } \alpha\,.\,\alpha = \alpha^2 = b_1$$

Let $\alpha$, $\beta$ be the roots of $x^2 - ax + b = 0$, then $\alpha + \beta = a$, $\alpha\beta = b$. Now

$$2(b + b_1) = 2b + b_1 = 2\alpha\beta + 2\alpha^2 = 2\alpha(\alpha + \beta) = 2.\frac{a_1}{2} \times a = aa_1.$$

Hence $2(b + b_1) = aa_1$.

**SA.6** If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root in common, find the relation between a, b and c.

**Sol.** Let $\alpha$ be the common root of the two equations. Then,

$$a\alpha^2 + b\alpha + c = 0$$

and $\quad b\alpha^2 + c\alpha + a = 0$

Solving these equations by cross-multiplication, we get

$$\frac{\alpha^2}{ab - c^2} = \frac{\alpha}{cb - a^2} = \frac{1}{ac - b^2}$$

$$\Rightarrow \quad \alpha = \frac{ab - c^2}{cb - a^2} \text{ or } \alpha = \frac{cb - a^2}{ac - b^2}$$

$$\Rightarrow \quad \frac{ab - c^2}{cb - a^2} = \frac{cb - a^2}{ac - b^2}$$

$\Rightarrow \quad (cb - a^2)^2 = (ab - c^2)(ac - b^2)$

$\Rightarrow \quad a(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow \quad a = 0$ or $a^3 + b^3 + c^3 = 3abc$.

This is the required relation between a, b and c.

**SA.7** If each pair of the three equation $x^2 + p_1x + q_1 = 0$, $x^2 + p_2x + q_2 = 0$ and $x^2 + p_3x + q_3 = 0$ have a common root, then prove that $p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$.

**Sol.** The three equations are :

$x^2 + p_1x + q_1 = 0$ ......(1)

$x^2 + p_2x + q_2 = 0$ ......(2)

$x^2 + p_3x + q_3 = 0$ ......(3)

Since each pair has a common root, the roots of (1), (2) and (3) can be taken as $\alpha$, $\beta$; $\beta$, $\gamma$ and $\gamma$, $\alpha$ respectively.

$\therefore \quad \alpha + \beta = -p_1$, $\alpha\beta = q_1$ ......(4)

$\beta + \gamma = -p_2$, $\beta\gamma = q_2$ ......(5)

and $\quad \gamma + \alpha = -p_3$, $\gamma\alpha = q_3$ ......(6)

Now, $\quad (\alpha + \beta) - (\beta + \gamma) = (-p_1 + p_2)$

$\Rightarrow \quad \alpha - \gamma = (p_2 - p_1)$

$\Rightarrow \quad (\alpha - \gamma)^2 = (p_2 - p_1)^2$

$\Rightarrow \quad (\alpha + \gamma)^2 - 4\alpha\gamma = (p_2 - p_1)^2$

$\Rightarrow \quad (-p_3)^2 - 4q_3 = (p_2 - p_1)^2$

$\Rightarrow \quad (p_2 - p_1)^2 = p_3^2 - 4q_3$ ......(7)

Similarly, $(p_3 - p_1)^2 = p_2^2 - 4q_2$ ......(8)

and $(p_2 - p_3)^2 = p_1^2 - 4q_1$ ......(9)

Adding (7), (8) and (9), we get

$$(p_2 - p_1)^2 + (p_3 - p_1)^2 + (p_2 - p_3)^2 = (p_3^2 - 4q_3) + (p_2^2 - 4q_2) + (p_1^2 - 4q_1)$$

$\Rightarrow \quad p_1^2 + p_2^2 + p_3^2 + 4(q_1 + q_2 + q_3) = 2(p_1p_2 + p_2p_3 + p_3p_1)$.

**SA.8** Find m and n is order that the equation $mx^2 + 5x + 2 = 0$ and $3x^2 + 10x + n = 0$ may have both the roots common.

**Sol.** The given equations are :

$mx^2 + 5x + 2 = 0$ ......(1)

$3x^2 + 10x + n = 0$ ......(2)

Since, equation (1) and (2) have both the roots common. Therefore

$$\frac{m}{3} = \frac{5}{10} = \frac{2}{n} \quad ......(3) \qquad \left[\because \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}\right]$$

I II III

From I and II members of (3), we get

$$m = \frac{15}{10} = \frac{3}{2}$$

From II and III members of (3), we get

$$n = \frac{20}{5} = 4$$

Hence, $m = \frac{3}{2}$ and $n = 4$.

**SA.9** Find the condition that the roots of the equation $ax^2 + bx + c = 0$ may be in the ratio p : q.

**Sol.** Let the roots of $ax^2 + bx + c = 0$ be $p\alpha$, $q\alpha$.

Then sum of the roots $= p\alpha + q\alpha = -\frac{b}{a}$

$\Rightarrow \quad (p + q)\alpha = -\frac{b}{a}$ .......(1)

and product of the roots $= p\alpha \,.\, q\alpha = \frac{c}{a}$

$\Rightarrow \quad pq\,\alpha^2 = \frac{c}{a}$ .......(2)

To find the condition, we have eliminate $\alpha$ between (1) and (2).

For this, on squaring (1), we get

$\Rightarrow \quad (p + q)^2\,\alpha^2 = \frac{b^2}{a^2}$ .......(3)

Dividing (3) by (2), we get

$$\frac{(p+q)^2}{pq} = \frac{b^2}{a^2} \cdot \frac{a}{c}$$

$\Rightarrow$ $ac\,(p + q)^2 = b^2pq$,

Which is the required condition.

**SA.10** Find the conditions that the roots of the equation $ax^2 + bx + c = 0$ with real coefficients may be

(a) both +ve

(b) one (with greater numerical value) positive and other –ve.

**Sol.** Let the roots be $\alpha$ and $\beta$, then

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Note that in this case roots must be real i.e., $b^2 - 4ac \geq 0$.

(a) When $\alpha$, $\beta$ are both +ve, then

$\alpha + \beta > 0$ and $\alpha\beta > 0$.

$\Rightarrow$ $-\frac{b}{a} > 0$ and $\frac{c}{a} > 0$.

$\Rightarrow$ $\frac{b}{a} < 0$ and $\frac{c}{a} > 0$.

$\Rightarrow$ a and b are opposite signs and c and a are of the same sign.

(b) When $\alpha > 0$ and $\beta < 0$, then

$\alpha + \beta > 0$ and $\alpha\beta < 0$ $\quad$ $[\because |\alpha| > |\beta|, \therefore \alpha + \beta = |\alpha| - (-\beta) = |\alpha| - |\beta| > 0]$

$\Rightarrow$ $-\frac{b}{a} > 0$ and $\frac{c}{a} < 0$.

$\Rightarrow$ $\frac{b}{a} < 0$ and $\frac{c}{a} < 0$.

$\Rightarrow$ b and a are of opposite signs and also c, a are of opposite signs

$\Rightarrow$ b and c are of same sign, opposite to that of a.

**SA.11** If the equation $x^2 + ax + bc = 0$ and $x^2 + bx + ca = 0$ have a common root, show that their other roots are given by the equation : $x^2 + cx + ab = 0$.

**Sol.** The given equation are

$x^2 + ax + bc = 0$ .........(1)

and $x^2 + bx + ca = 0$ .........(2)

Let $\alpha$ be their common root.

$\therefore$ By def.,

$\alpha^2 + a\alpha + bc = 0$ .........(3)

and $\alpha^2 + b\alpha + ca = 0$ .........(4)

Solving, by cross-multiplication, we get :

$$\frac{\alpha^2}{a^2c - b^2c} = \frac{\alpha}{bc - ca} = \frac{1}{b - c}$$

$$\Rightarrow \quad \frac{\alpha^2}{c(a^2-b^2)} = \frac{\alpha}{-c(a-b)} = \frac{1}{-(a-b)} \quad .........(5)$$

Taking first two members,

$$\alpha = \frac{c(a^2-b^2)}{-c(a-b)} = -(a+b) \quad .........(6)$$

Taking last two members,

$$\alpha = \frac{-c(a-b)}{-(a-b)} = c \quad .........(7)$$

From (6) and (7),

$$-(a+b) = c \quad .........(8)$$

Thus c is the common root of (1) and (2).

Product of roots of (1) = bc.

If one root is c, then the other root = $\frac{bc}{c}$ = b.

Again product of roots of (2) = ca.

If one root is c, then the other root = $\frac{ca}{c}$ = a.

∴ The equation whose roots are a, b is :

$x^2 - (a + b)x + ab = 0$ [$x^2 - Sx + P = 0$]

⇒ $x^2 + cx + ab = 0$, [using (8)]

Which is true.