Problem Based on Symmetric Functions and Formation of Quadratic Equation

VERY SHORT ANSWER TYPE QUESTIONS :

VSA.1 Find the quadratic equation whose roots are

(a) $2 + 3i$ and $1 - 7i$ (b) $\frac{k-1}{k+1}$ and $-\frac{k+1}{k-1}$

Sol.

(a) Given roots are $2 + 3i$ and $1 - 7i$

$\therefore$ sum of roots = $(2 + 3i) + (1 - 7i) = 3 - 4i$

product of roots = $(2 + 3i)(1 - 7i) = 2 - 14i + 3i - 21i^2 = 23 - 11i$

$\therefore$ the required equation is

$x^2 - (3 - 4i)x + 23 - 11i = 0.$

(b) Given roots are $\left(\frac{k-1}{k+1}\right)$ and $-\left(\frac{k+1}{k-1}\right)$

$\therefore$ sum of roots = $\left(\frac{k-1}{k+1}\right) - \left(\frac{k+1}{k-1}\right) = \frac{(k-1)^2 - (k+1)^2}{(k+1)(k-1)} = -\frac{4k}{k^2-1}$

product of roots = $\left(\frac{k-1}{k+1}\right)\left(-\frac{k+1}{k-1}\right) = -1$

$\therefore$ the required equation is

$x^2 - \left(-\frac{4k}{k^2-1}\right)x - 1 = 0$

$\Rightarrow$ $x^2 + \frac{4k}{k^2-1}x - 1 = 0$

$\Rightarrow$ $(k^2 - 1)x^2 + 4kx - (k^2 - 1) = 0$

VSA.2 Find the quadratic equation with real coefficients, one of whose roots is $-7 + 8i$.

Sol. $\because$ The non real roots of a quadratic equation with real coefficient occurs in conjugate pairs, therefore the other root is to be $-7 - 8i$.

$\therefore$ sum of roots = $(-7 + 8i) + (-7 - 8i) = -14$

and product of roots = $(-7 + 8i) + (-7 - 8i) = (-7)^2 + (8)^2 = 49 + 64 = 113.$

Hence required equation is

$x^2 - (-14)x + 113 = 0$

$\Rightarrow$ $x^2 + 14x + 113 = 0$

VSA.3 If α and β are the roots of the quadratic equation $3x^2 - 5x - 8 = 0$. Then find the value of

(a) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (b) $\alpha\beta^2 + \beta\alpha^2$ (c) $\alpha^2 + \beta^2$ (d) $\alpha^3 + \beta^3$

(e) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (f) $\alpha^4\beta^5 + \alpha^5\beta^4$ (g) $\sqrt{\alpha} + \sqrt{\beta}$

Sol. We have,

$3x^2 - 5x - 8 = 0$ ....(1)

$\because$ α, β are the roots of the equation (1) so

$\alpha + \beta = \frac{5}{3}$ $\alpha\beta = -\frac{8}{3}$

(a) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} = \dfrac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$

$$= \frac{\left(\frac{5}{3}\right)^2 - 2\left(-\frac{8}{3}\right)}{\left(-\frac{8}{3}\right)^2} = \frac{\frac{25}{9}+\frac{16}{3}}{\frac{64}{9}} = \frac{73}{64}$$

(b) $\alpha\beta^2 + \beta\alpha^2 = \alpha\beta(\alpha + \beta)$

$$= \left(-\frac{8}{3}\right)\left(\frac{5}{3}\right) = -\frac{40}{9}$$

(c) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{5}{3}\right)^2 - 2\left(-\frac{8}{3}\right) = \frac{25}{9} + \frac{16}{3} = \frac{73}{9}$$

(d) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$

$$= \left(\frac{5}{3}\right)^3 - 3\left(-\frac{8}{3}\right)\left(\frac{5}{3}\right) = \frac{125}{27} + 3 \times \frac{40}{9}$$

$$= \frac{125}{27} + \frac{40}{3} = \frac{125+360}{27} = \frac{485}{27}$$

(e) $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$

$$= \frac{\left(\frac{5}{3}\right)^3 - 3\left(-\frac{8}{3}\right)\left(\frac{5}{3}\right)}{\left(-\frac{8}{3}\right)} = \frac{\frac{125}{27}+\frac{40}{3}}{-\frac{8}{3}}$$

$$= \frac{\frac{485}{27}}{-\frac{8}{3}} = -\frac{485}{72}$$

(f) $\alpha^4\beta^5 + \alpha^5\beta^4 = \alpha^4\beta^4\,(\beta + \alpha) = (\alpha\beta)^4\,(\alpha + \beta)$

$$= \left(\frac{5}{3}\right)\left(-\frac{8}{3}\right)^4 = \frac{5}{3}\left(\frac{4096}{81}\right) = \frac{20480}{243}$$

(g) $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{(\sqrt{\alpha}+\sqrt{\beta})^2} = \sqrt{\alpha+\beta+2\sqrt{\alpha\beta}}$

$$= \sqrt{\left(\frac{5}{3}\right) + \sqrt{-\frac{8}{3}}} = \sqrt{\frac{5}{3} + i\sqrt{\frac{8}{3}}}$$

Problem Based on Symmetric Functions and Formation of Quadratic Equation

SHORT ANSWER TYPE QUESTIONS :

SA.1 If α, β are the roots (assumed non zero) of $3x^2 - 4x + 1 = 0$, form the equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.

Sol. The roots of $3x^2 - 4x + 1 = 0$ are α, β.

$\therefore \quad \alpha + \beta = -\frac{(-4)}{3} = \frac{4}{3}$ and $\alpha\beta = \frac{1}{3}$.

Let $\quad S = \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ are $P = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha}$.

$$\therefore \quad S = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{\left(\frac{4}{3}\right)^3 - 3\left(\frac{1}{3}\right)\left(\frac{4}{9}\right)}{\frac{1}{3}} = \frac{28}{9}$$

and $\quad P = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = \frac{1}{3}$.

$\therefore$ The equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$ is $x^2 - Sx + P = 0$

or $\quad x^2 - \frac{28}{9}x + \frac{1}{3} = 0$

or $\quad 9x^2 - 28x + 3 = 0$.

SA.2 If α, β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.

Sol. The roots of $ax^2 + bx + c = 0$ are α, β.

$\therefore \quad \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Let $\quad S = (\alpha^2 + \beta^2) + (\alpha^{-2} + \beta^{-2})$ and $P = (\alpha^2 + \beta^2)(\alpha^{-2} + \beta^{-2})$.

$$\therefore \quad S = (\alpha^2 + \beta^2) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = (\alpha^2 + \beta^2) + \left(\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}\right)$$

$$= (\alpha^2 + \beta^2) + \left(\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}\right)$$

$$= (\alpha^2 + \beta^2)\left(1 + \frac{1}{\alpha^2\beta^2}\right)$$

$= [(\alpha + \beta)^2 - 2\alpha\beta] \left[\left(1 + \frac{1}{(\alpha\beta)^2}\right)\right]$

$= \left[\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right] \left[\left(1 + \frac{1}{\left(\frac{c}{a}\right)^2}\right)\right]$

$= \left(\frac{b^2}{a^2} - \frac{2c}{a}\right) \left(1 + \frac{a^2}{c^2}\right) = \left(\frac{b^2 - 2ac}{a^2}\right) . \left(\frac{c^2 + a^2}{c^2}\right) = \frac{(b^2 - 2ac)(a^2 + c^2)}{a^2c^2}$

and $P = (\alpha^2 + \beta^2)\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = (\alpha^2 + \beta^2)\left(\frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}\right)$

$= \frac{(\alpha^2 + \beta^2)^2}{\alpha^2\beta^2} = \frac{[(\alpha + \beta)^2 - 2\alpha\beta]^2}{(\alpha\beta)^2}$

$= \frac{\left[\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right]^2}{\left(\frac{c}{a}\right)^2} = \left(\frac{b^2}{a^2} - \frac{2c}{a}\right)^2 \times \frac{a^2}{c^2} = \frac{(b^2 - 2ac)^2}{a^2c^2}.$

$\therefore$ The equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$ is $x^2 - Sx + P = 0$.

or $x - \frac{(b^2 - 2ac)(a^2 + c^2)}{a^2c^2}x + \frac{(b^2 - 2ac)^2}{a^2c^2} = 0$

or $a^2c^2x^2 - (b^2 - 2ac)(a^2 + c^2)x + (b^2 - 2ac)^2 = 0.$

SA.3 If α, β are the roots of the equation $ax^2 + bx + c = 0$, form a quadratic equation whose roots are $(a\alpha + b)^{-3}$ and $(a\beta + b)^{-3}$.

Sol. Here, $\alpha + \beta = -\frac{b}{a}$ ....(1)

and $\alpha\beta = \frac{c}{a}$ ....(2)

Since α is a root of $ax^2 + bx + c = 0$,

$\therefore$ $a\alpha^2 + b\alpha + c = 0$

$\Rightarrow$ $\alpha(a\alpha + b) = -c$

$\Rightarrow$ $a\alpha + b = -\frac{c}{\alpha}$ ....(3)

Similarly, we get

$a\beta + b = -\frac{c}{\beta}$ ....(4)

$\therefore$ Sum of the roots = $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$

$$= \left(-\frac{c}{\alpha}\right)^{-3} + \left(-\frac{c}{\beta}\right)^{-3} = \frac{b^3 - 3abc}{a^3b^3}$$

and product of the roots = $(a\alpha + b)^{-3} (a\beta + b)^{-3}$

$$= \left(-\frac{c}{\alpha}\right)^{-3} \left(-\frac{c}{\beta}\right)^{-3} = \frac{\alpha^3}{-c^3}.\frac{\beta^3}{-c^3}$$

$$= \frac{(\alpha\beta)^3}{c^6} = \frac{\left(\frac{c}{a}\right)^3}{c^6} = \frac{1}{a^3c^3}$$

Hence, the required equation is

$$x^2 - \left(\frac{b^3 - 3abc}{a^3c^3}\right)x + \frac{1}{a^3c^3} = 0$$

or $a^3 c^3 x^2 - (b^3 - 3abc) x + 1 = 0$

SA.4 If α and β are roots of the equation $ax^2 + bx + c = 0$, then form an equation whose roots are $\frac{1}{a\alpha + b}$, $\frac{1}{a\beta + b}$.

Sol. Since α, β are the roots of the equation

$ax^2 + bx + c = 0$ ....(1)

$\therefore$ $\alpha + \beta = -\frac{b}{a}$ and $\alpha.\beta = \frac{c}{a}$ ....(2)

Now, we have to form an equation whose roots are $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$.

$$\text{Sum of roots} = \frac{1}{a\alpha + b} + \frac{1}{a\beta + b} = \frac{a\beta + b + a\alpha + b}{(a\alpha + b)(a\beta + b)}$$

$$= \frac{2b + a(\alpha + \beta)}{a^2\alpha\beta + ab\alpha + ab\beta + b^2} = \frac{2b + a(\alpha + \beta)}{a^2\alpha\beta + ab(\alpha + \beta) + b^2}$$

$$= \frac{2b + a\left(-\frac{b}{a}\right)}{a^2\left(\frac{c}{a}\right) + ab\left(-\frac{b}{a}\right) + b^2} = \frac{b}{ac - b^2 + b^2} = \frac{b}{ac}$$ [using (2)]

$$\text{Product of roots} = \left(\frac{1}{a\alpha + b}\right)\left(\frac{1}{a\beta + b}\right) = \frac{1}{a^2\alpha\beta + ab(\alpha + \beta) + b^2}$$

$$= \frac{1}{ac - b^2 + b^2} = \frac{1}{ac}$$

The required equation will be

x^2 – (Sum of roots) x + Product of roots = 0

$\Rightarrow \quad x^2 - \frac{b}{ac}x + \frac{1}{ac} = 0$

$\Rightarrow \quad acx^2 - bx + 1 = 0,\ ac \neq 0.$

SA.5 If α and β are roots of the quadratic equation $px^2 + qx + r = 0$, then find the equation whose roots are $\sqrt{\frac{\alpha}{\beta}}$ and $\sqrt{\frac{\beta}{\alpha}}$.

Sol. Since α, β are the roots of the equation

$px^2 + qx + r = 0$

$\therefore \quad \alpha + \beta = -\frac{q}{p}$ and $\alpha.\beta = \frac{r}{p}$

Now, we have to form an equation whose roots are $\sqrt{\frac{\alpha}{\beta}}$ and $\sqrt{\frac{\beta}{\alpha}}$.

$$\text{Sum of roots} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = \frac{\alpha+\beta}{\sqrt{\alpha\beta}}$$

$$= \frac{-\frac{q}{p}}{\sqrt{\frac{r}{p}}} = \frac{-q}{\sqrt{rp}}$$

$$\text{Product of roots} = \sqrt{\frac{\alpha}{\beta}}.\sqrt{\frac{\beta}{\alpha}} = \sqrt{\frac{\alpha\beta}{\alpha\beta}} = 1$$

The required equation will be

x^2 – (Sum of roots)x + Product of roots = 0

$\Rightarrow \quad x^2 - \left(-\frac{q}{\sqrt{rp}}\right)x + 1 = 0$

$\Rightarrow \quad \sqrt{pr}\,x^2 - qx + \sqrt{pr} = 0,\ \sqrt{pr} \neq 0.$

SA.6 If α, β are the roots of the equation $ax^2 + bx + c = 0$ such that $\alpha > \beta$, then find the values of :

(a) $\alpha^2 - \beta^2$ (b) $\alpha^3 - \beta^3$.

Sol. α, β are the roots of $ax^2 + bx + c = 0$.

$\therefore \quad \alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(a) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$ ....(1)

Now, $\alpha - \beta = \sqrt{(\alpha-\beta)^2} = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$ $[\because \alpha > \beta \Rightarrow \alpha - \beta = 0]$

$$= \sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)} = \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}$$

$$= \sqrt{\frac{b^2 - 4ac}{a^2}} = \sqrt{\frac{b^2 - 4ac}{a}}$$

and $\alpha + \beta = -\frac{b}{a}$.

$\therefore$ (1) implies $\alpha^2 - \beta^2 = -\frac{b}{a} \times \sqrt{\frac{b^2 - 4ac}{a}} = -\frac{b\sqrt{b^2 - 4ac}}{a^2}$.

(b) $\alpha^3 - \beta^3 = (\alpha - \beta)\ (\alpha^2 + \alpha\beta + \beta^{2)}$ ....(1)

Now, $\alpha - \beta = \sqrt{(\alpha-\beta)^2} = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$$= \sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)} = \sqrt{\frac{b^2 - 4ac}{a}}$$

and $\alpha^2 + \alpha\beta + \beta^2 = (\alpha + \beta)^2 - \alpha\beta = \left(-\frac{b}{a}\right)^2 - \frac{c}{a} = \frac{b^2 - ac}{a^2}$

$\therefore$ (1) implies $\alpha^3 - \beta^3 = \sqrt{\frac{b^2 - 4ac}{a}} \times \frac{b^2 - ac}{a^2} = \frac{\sqrt{b^2 - 4ac}}{a^3}(b^2 - ac)$.

SA.7 If α, β are the roots of the equation $ax^2 + bx + c = 0$ such that α, β are non-zero and $\alpha > \beta$, then find the values of :

(a) $\frac{1}{\alpha} - \frac{1}{\beta}$ (b) $\frac{1}{\alpha^2} - \frac{1}{\beta^2}$.

Sol. α, β are the roots of $ax^2 + bx + c = 0$.

$\therefore$ $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(a) $\frac{1}{\alpha} - \frac{1}{\beta} = \frac{\beta - \alpha}{\alpha\beta} = -\frac{\alpha - \beta}{\alpha\beta} = -\frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{\alpha\beta}$ $[\because \alpha > \beta \Rightarrow \alpha - \beta > 0]$

$$= \frac{\sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)}}{\frac{c}{a}} = -\sqrt{\frac{b^2 - 4ac}{c}}.$$

(b) $$\frac{1}{\alpha^2} - \frac{1}{\beta^2} = \frac{\beta^2 - \alpha^2}{\alpha^2\beta^2} = \frac{(\beta-\alpha)(\beta+\alpha)}{(\alpha\beta)^2}$$

$$= -\frac{(\alpha-\beta)(\alpha+\beta)}{(\alpha\beta)^2} = -\frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}.(\alpha+\beta)}{(\alpha\beta)^2}$$

$$= \frac{\sqrt{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)}}{\left(\frac{c}{a}\right)^2}\left(-\frac{b}{a}\right) = \frac{b\sqrt{b^2-4ac}}{c^2}.$$

SA.8 If α and β are roots of the equation $x^2 - px + q = 0$, then find the value of

(a) $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 3\alpha\beta$

(b) $\alpha^2\left(\frac{\alpha^2}{\beta}+\beta\right) + \beta^2\left(\frac{\beta^2}{\alpha}+\alpha\right)$

(c) $\frac{1}{(p\alpha+q)^2} + \frac{1}{(p\beta+q)^2}$.

Sol. Since α, β are the roots of the equation

$x^2 - px + q = 0$ ....(1)

$\therefore$ $\alpha + \beta = p$ and $\alpha.\beta = q$ ....(2)

Now, $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = p^2 - 2q$ ....(3) [using (2)]

and $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = p^3 - 3pq$ ....(4) [using (2)]

(a) $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 3\alpha\beta$

$= \frac{1}{\alpha\beta}[\alpha^2 + \beta^2 + 2(\alpha+\beta) + 3\alpha^2\beta^2]$

$= \frac{1}{q}(p^2 - 2q + 2p + 3q^2)$ [using (2) and (3)]

(b) $$\alpha^2\left(\frac{\alpha^2}{\beta}+\beta\right) + \beta^2\left(\frac{\beta^2}{\alpha}+\alpha\right) = \alpha^2\left(\frac{\alpha^2+\beta^2}{\beta}\right) + \beta^2\left(\frac{\beta^2+\alpha^2}{\alpha}\right)$$

$$= \frac{(\alpha^2+\beta^2)(\alpha^3+\beta^3)}{\alpha\beta}$$

$= \frac{1}{q}(p^2 - 2q)(p^3 - 3pq)$ [using (3) and (4)]

(c) $\dfrac{1}{(p\alpha+q)^2} + \dfrac{1}{(p\beta+q)^2} = \dfrac{(p\beta+q)^2(p\alpha+q)^2}{[(p\alpha+q)\ (p\beta+q)]^2}$

$$= \frac{p^2(\alpha^2+\beta^2)+2pq(\alpha+\beta)+2q^2}{[p^2\alpha\beta+pq(\alpha+\beta)+q^2]^2}$$

$$= \frac{p^2\ (p^2-2q)+2pq(p)+2q^2}{[p^2q+pq(p)+q^2]^2} \qquad \text{[using (2) and (3)]}$$

$$= \frac{p^2+2q^2}{q^2(2p^2+q)^2}$$

SA.9 If α, β are the roots of the equation $3x^2 - 6x + 4 = 0$, evaluate

$$\left(\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}\right) + \left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 3\alpha\beta.$$

Sol. α, β are the roots of equation $3x^2 - 6x + 4 = 0$

$\Rightarrow \quad \alpha + \beta = \dfrac{6}{3} = 2$

and $\quad \alpha\beta = \dfrac{4}{3}$

$\therefore \quad \left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right) + \left(\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}\right) + 2\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right) + 3\alpha\beta.$

$$\frac{\alpha^3+\beta^3}{\alpha\beta} + \frac{\alpha^2+\beta^2}{\alpha\beta} + \frac{2(\alpha+\beta)}{\alpha\beta} + 3\alpha\beta.$$

$\Rightarrow \quad \dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta} + \dfrac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta} + \dfrac{2(\alpha+\beta)}{\alpha\beta} + 3\alpha\beta.$

$\Rightarrow \quad \dfrac{2^3-3\times\frac{4}{3}\times 2}{\frac{4}{3}} + \dfrac{4-2\times\frac{4}{3}}{\frac{4}{3}} + \dfrac{2\times 2}{\frac{4}{3}} + 3 \times \dfrac{4}{3}.$

$\Rightarrow \quad \dfrac{8-8}{\frac{4}{3}} + \dfrac{12-8}{4} + \dfrac{12}{4} + 4$

$\Rightarrow \quad 0 + 1 + 3 + 4 = 8.$

Problem Based on Symmetric Functions and Formation of Quadratic Equation

SA.10 If α and β are the roots of $x^2 - p(x + 1) - c = 0$, show that $(\alpha + 1)(\beta + 1) = 1 - c$. Hence prove that $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1$.

Sol. The given equation is

$$x^2 - px - (p + c) = 0.$$

$\therefore$ $\alpha + \beta = p$ and $\alpha\beta = -(p + c)$

So, $(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$

$$= -p - c + p + 1 = 1 - c \quad ......(1)$$

Now, $$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} \quad \text{[using (1)]}$$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha}$$

$$= \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1.$$