Problem Based on Relation Between Roots and Coefficients

VERY SHORT ANSWER TYPE QUESTIONS :

VSA.1 Find the sum and product of the roots of the following equations :

(a) $8x^2 - 7x + 15 = 0$ (b) $x^2 + 19 = 0$

(c) $\sqrt{3}x^2 - \frac{11}{2}x + \frac{3\sqrt{3}}{4} = 0$ (d) $(2 + i)x^2 + ix + 7 - i = 0$.

Sol.

(a) We have,

$$8x^2 - 7x + 15 = 0$$

$\therefore$ Sum of roots $= -\frac{\text{coeff. of } x}{\text{coeff. of } x^2} = -\frac{-7}{8} = \frac{7}{8}$.

and product of roots $= \frac{\text{constant term}}{\text{coeff. of } x^2} = \frac{15}{8}$.

(b) We have,

$$x^2 + 19 = 0 \quad \text{i.e., } x^2 + 0.x + 19 = 0$$

$\therefore$ Sum of roots $= -\frac{\text{coeff. of } x}{\text{coeff. of } x^2} = -\frac{0}{1} = 0$.

and product of roots $= \frac{\text{constant term}}{\text{coeff. of } x^2} = \frac{19}{1} = 19$.

(c) We have,

$$\sqrt{3}x^2 - \frac{11}{2}x + \frac{3\sqrt{3}}{4} = 0$$

$\therefore$ Sum of roots $= -\frac{\text{coeff. of } x}{\text{coeff. of } x^2} = -\frac{-11/2}{\sqrt{3}} = \frac{11}{2\sqrt{3}}$.

and product of roots $= \frac{\text{constant term}}{\text{coeff. of } x^2} = \frac{3\sqrt{3}/4}{\sqrt{3}} = \frac{3}{4}$.

(d) We have,

$$(2 + i)x^2 + ix + 7 - i = 0.$$

$\therefore$ Sum of roots $= -\frac{\text{coeff. of } x}{\text{coeff. of } x^2} = -\frac{i}{2+i} = -\frac{i}{2+i} \times \frac{2-i}{2-i}$

$$= \frac{-2i + i^2}{4+1} = -\frac{1}{5} - \frac{2}{5}i.$$

and product of roots $= \frac{\text{constant term}}{\text{coeff. of } x^2} = \frac{7-i}{2+i} = \frac{7-i}{2+i} \times \frac{2-i}{2-i}$

$$= \frac{14 - 7i - 2i - 1}{4+1} = \frac{13}{5} - \frac{9}{5}i.$$

Problem Based on Relation Between Roots and Coefficients

VSA.2 For the quadratic equation $(k - 1)x^2 = kx - 1$, find k, so that :

(a) one root is –3.

(b) the sum of the roots is 2.

(c) the product of the roots is –3.

(d) the roots are equal.

(e) the roots are numerically equal but opposite in sign.

Sol. We have,

$(k - 1)x^2 = kx - 1$

$\Rightarrow$ $(k - 1)x^2 - kx + 1 = 0$ ....(1)

(a) one roots is –3.

$\therefore$ –3 satisfies (1).

$\therefore$ $(k - 1)(-3)^2 - k(-3) + 1 = 0$

$\Rightarrow$ $9k - 9 + 3k + 1 = 0$

$\Rightarrow$ $12k = 8$

$\Rightarrow$ $k = \frac{2}{3}$.

(b) Sum of roots is 2.

$\therefore$ (1) $-\frac{-k}{k-1} = 2$ $\left[\because \text{Sum of roots} = \frac{\text{coeff. of } x}{\text{coeff. of } x^2}\right]$

$\Rightarrow$ $\frac{k}{k-1} = 2$

$\Rightarrow$ $k = 2k - 2$

$\Rightarrow$ $k = 2$.

(c) Product of roots is –3.

$\therefore$ (1) $\frac{1}{k-1} = -3$ $\left[\because \text{Product of roots} = \frac{\text{constant term}}{\text{coeff. of } x^2}\right]$

$\Rightarrow$ $1 = -3k + 3$

$\Rightarrow$ $3k = 2$

$\Rightarrow$ $k = \frac{2}{3}$.

(d) Roots are equal.

$\therefore$ Disc. = 0

$\therefore$ $(-k)^2 - 4(k - 1)(1) = 0$

$\Rightarrow$ $k^2 - 4k + 4 = 0$

$\Rightarrow$ $(k - 2)^2 = 0$

$\Rightarrow$ $k - 2 = 0$

$\Rightarrow$ $k = 2$

(e) Roots are numerically equal but opposite in sign.

$\therefore$ Sum of roots = 0

$\therefore$ (1) $-\frac{-k}{k-1} = 0$

$\Rightarrow$ $k = 0$

Problem Based on Relation Between Roots and Coefficients

VSA.3 If α, β are the roots of $x^2 - p(x + 1) - c = 0$, show that :

$$(\alpha + 1)(\beta + 1) = 1 - c$$

Sol. The given equation can be written as :

$$x^2 - px - (p + c) = 0$$

If α, β be its roots, then $\alpha + \beta = p$ and $\alpha\beta = -(p + c)$ ......(1)

Now $(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$

$= -(p + c) + (p) + 1$ [using (1)]

$= 1 - c$, which is true.

VSA.4 Form the equations whose roots are :

(a) $\frac{3-\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}$ (b) 7i, 2i

(c) $\frac{i}{4}, -\frac{i}{4}$ (d) 3 – 4i, 2 + 3i

Sol.

(a) Given roots are $\frac{3-\sqrt{2}}{2}$ and $\frac{3+\sqrt{2}}{2}$

Sum of roots $= \frac{3-\sqrt{2}}{2} + \frac{3+\sqrt{2}}{2} = 3$

And product of roots $= \left(\frac{3-\sqrt{2}}{2}\right)\left(\frac{3+\sqrt{2}}{2}\right) = \frac{9-2}{4} = \frac{7}{4}$

Therefore, the required equation is

x^2 – (sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - 3x + \frac{7}{4} = 0$

$\Rightarrow$ $4x^2 - 12x + 7 = 0$

(b) Given roots are 7i, 2i

Sum of roots = 7i + 2i = 9i

And product of roots = (7i) (2i) = $14i^2 = -14$ [$\because i^2 = -1$]

Therefore, the required equation is

x^2 – (sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - (9i)x + (-14) = 0$

$\Rightarrow$ $x^2 - 9ix - 14 = 0$

(c) Given roots are $\frac{i}{4}, \frac{i}{4}$

Sum of roots $= \frac{i}{4} - \frac{i}{4} = 0$

And product of roots $= \left(\frac{i}{4}\right)\left(-\frac{i}{4}\right) = -\frac{i^2}{16} = \frac{1}{16}$ [$\because i^2 = -1$]

Therefore, the required equation is

x^2 – (sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - (0)\,x + \frac{1}{16} = 0$

$\Rightarrow$ $16x^2 + 1 = 0$

(d) Given roots are 3 – 4i and 2 + 3i

Sum of roots = 3 – 4i + 2 + 3i = 5 – i

And product of roots = (3 – 4i) (2 + 3i) = 6 + 9i – 8i – $12i^2$

= 6 + i + 12 = 18 + i

The required equation is

x^2 – (sum of roots) x + Product of roots = 0

$\Rightarrow$ $x^2 - (5 - i)\,x + 18 + i = 0$

VSA.5 Prove that the roots of the quadratic equation $ax^2 + bx + a = 0$ are reciprocals of each other.

Sol. The given equation is

$ax^2 + bx + a = 0$

Let the roots is given quadratic equation are α and β.

Product of two roots = $\alpha.\beta = \frac{a}{a} = 1$

Therefore, the roots are reciprocal.

VSA.6 Find the value of 'a' for which the roots α, β of the equation $x^2 - 6x + a = 0$ satisfy the relation $3\alpha + 2\beta = 20$.

Sol. Since, α, β are the roots of $x^2 - 6x + a = 0$

$\therefore$ $\alpha + \beta = 6$ ....(1)

and $\alpha \times \beta = a$ ....(2)

Now $3\alpha + 2\beta = 20$

$\Rightarrow$ $\alpha + 2(\alpha + \beta) = 20$

$\Rightarrow$ $\alpha + 2 \times 6 = 20$

$\Rightarrow$ $\alpha = 8$

Putting $\alpha = 8$ in (1), we obtain $\beta = -2$.

Substituting the values of α and β in (2), we get a = –16.

VSA.7 If the roots of the equation $x^2 - \ell x + m = 0$ differ by 1, then prove that $\ell^2 = 4m + 1$.

Sol. Let α, β be the roots of the equation $x^2 - \ell x + m = 0$. Then

$\alpha + \beta = \ell$

and $\alpha\beta = m$

We have,

$\alpha - \beta = 1$

$\Rightarrow$ $(\alpha - \beta)^2 = 1$

$\Rightarrow$ $(\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow$ $\ell^2 - 4m = 1$

$\Rightarrow$ $\ell^2 = 4m + 1$

Problem Based on Relation Between Roots and Coefficients

SHORT ANSWER TYPE QUESTIONS :

SA.1 The roots r_1 and r_2 of a quadratic equation $x^2 + kx + 12 = 0$ are such that $r_1 - r_2 = 1$. Find k.

Sol. The given equation is

$x^2 + kx + 12 = 0$ ....(1)

It roots are r_1 and r_2.

∴ $r_1 + r_2 = -k$ ....(2)

and $r_1r_2 = 12$ ....(3)

Also $r_1 - r_2 = 1$ ....(4) [Given]

Now, $(r_1 - r_2)^2 = (r_1 + r_2)^2 - 4r_1r_2$

∴ $(1)^2 = (-k)^2 - 4(12)$ [using (1), (2) & (3)]

⇒ $1 = k^2 - 48$

⇒ $k^2 = 1 + 48 = 49$

⇒ $k = \pm 7$.

Hence $k = \pm 7$.

SA.2 For what value of m ($m \neq -1$) will the equation $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ has roots equal in magnitude but opposite in sign.

Sol. We have,

$$\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$$

⇒ $(m + 1)x^2 - b(m + 1)x = a(m - 1)x - (m - 1)x$

⇒ $(m + 1)x^2 - (bm + b + am - a)x + (m - 1)c = 0$

Since roots are equal in magnitude but opposite in signs, the sum of roots must be zero.

∴ $-\frac{(bm + b + am - a)}{m+1} = 0$ $\left[\because \text{sum of roots} = -\frac{\text{coeff. of } x}{\text{coeff. of } x^2}\right]$

⇒ $bm + b + am - a = 0$

⇒ $m = \frac{a-b}{a+b}$.

SA.3 Solve the equation $x^2 + px + 45 = 0$, given that the square of the difference of its roots is equal to 144.

Sol. The given equation is

$x^2 + px + 45 = 0$ ....(1)

Let its roots be α and β, then

$\alpha + \beta = -p$ and $\alpha\beta = 45$.

According to given,

$(\alpha - \beta)^2 = 144$

⇒ $(\alpha + \beta)^2 - 4\alpha\beta = 144$

⇒ $(-p)^2 - 4.45 = 144$

⇒ $p^2 = 324$

⇒ $p = \pm 18$.

Case I : $p = 18$, then we get

$x^2 + 18x + 45 = 0$

$\Rightarrow (x + 3)(x + 15) = 0$

$\Rightarrow x = -3, -15$

Case II : $p = -18$, then we get

$x^2 - 18x + 45 = 0$

$\Rightarrow (x - 3)(x - 15) = 0$

$\Rightarrow x = 3, 15$

Hence the roots of the given equation are $-3, -15$ or $3, 15$.

SA.4 Find the value of k for which the roots α, β of the equation $5x^2 + (2k + 1)x + (k - 2) = 0$ satisfy the relation $2\alpha + 5\beta = 1$.

Sol. We have,

$$5x^2 + (2k + 1)x + k - 2 = 0$$

$$\therefore \quad \alpha + \beta = -\frac{2k+1}{5} \quad .....(1)$$

$$\text{and} \quad \alpha\beta = \frac{k-2}{5} \quad .....(2)$$

$$\text{Also,} \quad 2\alpha + 5\beta = 1 \quad .....(3)$$

(1) × 2

$$\Rightarrow \quad 2\alpha + 2\beta = -\frac{4k+2}{5} \quad .....(4)$$

(3) − (4)

$$\Rightarrow \quad 3\beta = 1 + \frac{4k+2}{5} = \frac{4k+7}{5}$$

$$\Rightarrow \quad \beta = \frac{4k+7}{15}$$

$\therefore$ (1) implies

$$\alpha = -\frac{2k+1}{5} - \beta = -\frac{2k+1}{5} - \frac{4k+7}{15} = \frac{-6k-3-4k-7}{15}$$

$$= \frac{-10k-10}{15} = -\frac{2k+2}{3}.$$

Putting the values of α and β in (2), we get

$$\left(-\frac{2k+2}{3}\right)\left(\frac{4k+7}{15}\right) = \frac{k-2}{5}$$

$\Rightarrow -(2k + 2)(4k + 7) = 9(k - 2)$

$\Rightarrow -8k^2 - 14k - 8k - 14 = 9k - 18$

$\Rightarrow 8k^2 + 31k - 4 = 0$

$\Rightarrow (8k - 1)(k + 4) = 0$

$\Rightarrow k = \frac{1}{8}, -4.$

Problem Based on Relation Between Roots and Coefficients

SA.5 If one of the roots of the quadratic equation $ax^2 + bx + c = 0$ is the square of the other, then prove that $b^3 + a^2c + ac^2 = 3abc$.

Sol. The given equation is

$$ax^2 + bx + c = 0$$

Let one roots be α, then the other root = α^2. Then,

$$\alpha + \alpha^2 = -\frac{b}{a} \qquad ......(1)$$

and $$\alpha . \alpha^2 = \frac{c}{a}$$

$$\Rightarrow \quad \alpha^3 = \frac{c}{a} \qquad ......(2)$$

Cubing both sides of (1), we get

$$(\alpha + \alpha^2)^3 = -\left(\frac{b}{a}\right)^3$$

$$\Rightarrow \quad \alpha^3 + (\alpha^2)^3 + 3\alpha . \alpha^2 (\alpha + \alpha^2) = -\frac{b^3}{a^3} \qquad [\because (x + y)^3 = x^3 + y^3 + 3xy (x + y)]$$

$$\Rightarrow \quad \alpha^3 + \alpha^6 + 3\alpha^3 (\alpha + \alpha^2) = -\frac{b^3}{a^3} \qquad ......(3)$$

Substituting the value of α^3 and $\alpha + \alpha^2$ from (1) and (2) in (3), we get

$$\frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}$$

$$\Rightarrow \quad \frac{c}{a} + \frac{c^2}{a^2} - \frac{3bc}{a^2} = -\frac{b^3}{a^3}$$

Multiplying both sides by a^3, we get

$$a^2c + ac^2 - 3abc = -b^3$$

$$\Rightarrow \quad b^3 + a^2c + ac^2 = 3abc$$

SA.6 Show that the A.M. of the roots of $x^2 - 2ax + b^2 = 0$ is equal to the G.M. of the roots of the equation $x^2 - 2bx + a^2 = 0$ and vice-versa.

Sol. We have,

$$x^2 - 2ax + b^2 = 0 \qquad ....(1)$$

and $$x^2 - 2bx + a^2 = 0 \qquad ....(2)$$

Let α, β be the roots of (1) and γ, δ be the roots of (2).

Then, $\alpha + \beta = 2a$, and $\alpha\beta = b^2$ ....(3)

$\gamma + \delta = 2b$, and $\gamma\delta = a^2$ ....(4)

Now A.M. of α, β = $\frac{\alpha + \beta}{2} = \frac{2a}{2} = a$ [using (3)]

G.M. of γ, δ = $\sqrt{\gamma\delta} = \sqrt{a^2} = a$ [using (4)]

$\therefore$ A.M. of α, β = G.M. of γ, δ

Again, A.M. of $\gamma, \delta = \frac{\gamma+\delta}{2} = \frac{2b}{2} = b$ [using (4)]

Again, G.M. of $\alpha, \beta = \sqrt{\alpha\beta} = \sqrt{b^2} = b$ [using (3)]

$\therefore$ A.M. of γ, δ = G.M. of α, β.

SA.7 If the ratio of the roots of the equation $\ell x^2 + nx + n = 0$ is equal to p : q, prove that

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{\ell}} = 0$$

Sol. Let α, β be the roots of the equation $\ell x^2 + nx + n = 0$

Then $\alpha + \beta = -\frac{n}{\ell}$ and $\alpha\beta = \frac{n}{\ell}$ ......(1)

We have,

$$\frac{\alpha}{\beta} = \frac{p}{q}$$

$\therefore$ $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{\ell}} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{n}{\ell}}$ $\left[\text{using} : \frac{p}{q} = \frac{\alpha}{\beta}\right]$

$= \frac{\alpha+\beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{n}{\ell}} = \frac{-\frac{n}{\ell}}{\sqrt{\frac{n}{\ell}}} + \sqrt{\frac{n}{\ell}}$ [using (1)]

$= -\sqrt{\frac{n}{\ell}} + \sqrt{\frac{n}{\ell}} = 0$

SA.8 If α, β be the roots of $ax^2 + 2bx + c = 0$ and $\alpha + \delta, \beta + \delta$ be those of $Ax^2 + 2Bx + C = 0$, then prove that $\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$.

Sol. Since α, β are roots of $ax^2 + 2bx + c = 0$.

$\therefore$ $\alpha + \beta = -\frac{2b}{a}$ and $\alpha\beta = \frac{c}{a}$ ......(1)

Let $\alpha' = \alpha + \delta$ and $\beta' = \beta + \delta$

It is given that $\alpha' = \alpha + \delta$, $\beta' = \beta + \delta$ are the roots of the equation

$Ax^2 + 2Bx + C = 0$

$\therefore$ $\alpha' + \beta' = -\frac{2B}{A}$ and $\alpha'\beta' = \frac{C}{A}$ ......(2)

Now, $\alpha' = \alpha + \delta$ and $\beta' = \beta + \delta$

$\Rightarrow$ $\alpha' - \beta' = (\alpha + \beta) - (\beta + \alpha) = \alpha - \beta$

$\Rightarrow$ $(\alpha' - \beta')^2 = (\alpha - \beta)^2$

$\Rightarrow$ $(\alpha' + \beta')^2 - 4\alpha'\beta' = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow \qquad \dfrac{4B^2}{A^2} - \dfrac{4C}{A} = \dfrac{4b^2}{a^2} - \dfrac{4c}{a}$

$\Rightarrow \qquad \dfrac{B^2 - AC}{A^2} = \dfrac{b^2 - ac}{a^2}$

$\Rightarrow \qquad \dfrac{b^2 - ac}{B^2 - AC} = \left(\dfrac{a}{A}\right)^2.$

SA.9 If p be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\dfrac{(p+1)^2}{p} = \dfrac{b^2}{ac}$.

Sol. We have $ax^2 + bc + c = 0$. Let the roots of this equation be α and αp.

$\therefore \qquad \alpha + \alpha p = -\dfrac{b}{a}$ ....(1)

and $\qquad \alpha . \alpha p = \dfrac{c}{a}$ ....(2)

(1) $\Rightarrow \qquad \alpha(1 + p) = -\dfrac{b}{a}$

$\Rightarrow \qquad \alpha^2(1 + p)^2 = \dfrac{b^2}{a^2}$ ....(3)

(2) $\Rightarrow \qquad \alpha^2 p = \dfrac{c}{a}$ ....(4)

Dividing (3) by (4), we get

$$\frac{a^2(1+p)^2}{\alpha^2 p} = \frac{b^2/a^2}{c/a}$$

$\Rightarrow \qquad \dfrac{(1+p)^2}{p} = \dfrac{b^2}{ac}.$

SA.10 Given that α, β are the roots of the equation $\lambda(x^2 - x) + x + 5 = 0$. If λ_1, λ_2 are the values of λ for which the roots α, β are related by $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{5}$, find the values of

(a) $\dfrac{\lambda_1}{\lambda_2} + \dfrac{\lambda_2}{\lambda_1}$ and (b) $\dfrac{\lambda_1}{\lambda_2^2} + \dfrac{\lambda_2}{\lambda_1^2}$.

Sol. We have,

$\lambda(x^2 - x) + x + 5 = 0$

$\therefore \qquad \lambda x^2 - (\lambda - 1)x + 5 = 0$

$\therefore \qquad \alpha + \beta = \dfrac{\lambda - 1}{\lambda}$ and $\alpha\beta = \dfrac{5}{\lambda}$

Also $\qquad \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{5}$

$\therefore \quad \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{4}{5}$

$\Rightarrow \quad 5((\alpha + \beta)^2 - 2\alpha\beta) = 4\alpha\beta$

$\Rightarrow \quad 5(\alpha + \beta)^2 - 14\alpha\beta = 0$

$\Rightarrow \quad 5\left(\dfrac{\lambda-1}{\lambda}\right)^2 - 14\left(\dfrac{5}{\lambda}\right) = 0$

$\Rightarrow \quad (\lambda^2 - 2\lambda + 1) - 14\lambda = 0$

$\Rightarrow \quad \lambda^2 - 16\lambda + 1 = 0$

$\Rightarrow \quad \lambda_1 + \lambda_2 = 16$ and $\lambda_1 \lambda_2 = 1$.

(a) $\dfrac{\lambda_1}{\lambda_2} + \dfrac{\lambda_2}{\lambda_1} = \dfrac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2} = \dfrac{(\lambda_1+\lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2} = \dfrac{(16)^2-2(1)}{1} = 254.$

(b) $\dfrac{\lambda_1}{\lambda_2^2} + \dfrac{\lambda_2}{\lambda_1^2} = \dfrac{\lambda_1^3+\lambda_2^3}{\lambda_1^2\lambda_2^2} = \dfrac{(\lambda_1+\lambda_2)^3 - 3\lambda_1\lambda_2(\lambda_1+\lambda_2)}{(\lambda_1\lambda_2)^2} = \dfrac{(16)^3-3(1)(16)}{(1)^2} = 4096 - 48 = 4048.$

SA.11 If p, q be the roots of $3x^2 + 6x + 2 = 0$, form an equation whose roots are $\dfrac{-p^2}{q}, \dfrac{-q^2}{p}$.

Sol. Since p, q are the roots of

$3x^2 + 6x + 2 = 0$

$\therefore \quad p + q = -\dfrac{6}{3} = -2 \qquad ....(1)$

and $\quad pq = \dfrac{2}{3} \qquad ....(2)$

Here, $\quad S = \left(-\dfrac{p^2}{q}\right) + \left(-\dfrac{q^2}{p}\right) = -\left(\dfrac{p^2}{q} + \dfrac{q^2}{p}\right)$

$= -\dfrac{p^3+q^3}{pq} = -\dfrac{(p+q)^3 - 3pq(p+q)}{pq}$

$= -\dfrac{(-2)^3 - 3.\frac{2}{3}.(-2)}{\frac{2}{3}}$ [using (1) and (2)]

$= -(-8 + 4)\,.\,\dfrac{3}{2} = 6$

and $\quad P = \left(-\dfrac{p^2}{q}\right)\left(\dfrac{q^2}{p}\right) = pq = \dfrac{2}{3}.$ [using (2)]

$\therefore$ The required equation is

$x^2 - 6x + \dfrac{2}{3} = 0$ $[x^2 - Sx + P = 0]$

or $\quad 3x^2 - 18x + 2 = 0$

LONG ANSWER TYPE QUESTIONS :

LA.1 Find the number of quadratic equations which are unchanged by squaring their roots.

Sol. Let α and β be the roots of quadratic equation and α^2 and β^2 be the roots of another equation. By the question,

$$\alpha + \beta = \alpha^2 + \beta^2 \quad ......(1)$$

and $$\alpha\beta = \alpha^2\beta^2 \quad ......(2)$$

From (2),

$$\alpha\beta(1 - \alpha\beta) = 0$$

$\Rightarrow$ $\alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$.

(a) When $\alpha = 0$.

From (1), $\beta = \beta^2$

$\Rightarrow$ $\beta(1 - \beta) = 0$

$\Rightarrow$ $\beta = 0$ or 1.

Thus we have : $\alpha = 0$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$.

(b) When $\beta = 0$.

From (1), $\alpha = \alpha^2$

$\Rightarrow$ $\alpha(1 - \alpha) = 0$

$\Rightarrow$ $\alpha = 0$ or 1.

Thus we have : $\alpha = 0$, $\beta = 0$ and $\alpha = 1$, $\beta = 0$.

(c) When $\alpha\beta = 1$

From (1), $\alpha + \beta = \alpha^2 + \beta^2$

$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \alpha^2 + \frac{1}{\alpha^2}$$

$$\Rightarrow \quad \alpha + \frac{1}{\alpha} = \left(\alpha + \frac{1}{\alpha}\right)^2 - 2.$$

Putting $\alpha + \frac{1}{\alpha} = y$, we get

$$y = y^2 - 2$$

$\Rightarrow$ $y^2 - y - 2 = 0$

$\Rightarrow$ $(y - 2)(y + 1) = 0$

$\Rightarrow$ $y = 2$ or -1.

When $y = 2$, then $\alpha + \frac{1}{\alpha} = 2$

$$\alpha^2 - 2\alpha + 1 = 0$$

$\Rightarrow$ $(\alpha - 1)^2 = 0$

$\Rightarrow$ $\alpha = 1$.

When $y = -1$, then $\alpha + \frac{1}{\alpha} = -1$

$$\alpha^2 + \alpha + 1 = 0$$

$$\Rightarrow \quad \alpha = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \omega, \omega^2.$$

When $\alpha = 1,\ \alpha\beta = 1$

$\Rightarrow$ $\beta = 1$.

When $\alpha = \omega,\ \alpha\beta = 1$

$$\Rightarrow \quad \beta = \frac{1}{\alpha} = \frac{1}{\omega} = \frac{\omega^3}{\omega} = \omega^2.$$

When $\alpha = \omega^2,\ \alpha\beta = 1$

$$\Rightarrow \quad \beta = \frac{1}{\alpha} = \frac{1}{\omega^2} = \frac{\omega^3}{\omega^2} = \omega.$$

Thus $\alpha = 1,\ \beta = 1;\ \alpha = \omega,\ \beta = \omega^2$.

Hence there are four quadratic equations.

LA.2 Let α, β be two real numbers satisfying the following relations :

$$\alpha^2 + \beta^2 = 5,\ 3(\alpha^5 + \beta^5) = 11(\alpha^3 + \beta^3)$$

Find the quadratic equation having α and β as its roots.

Sol. We have,

$$3(\alpha^5 + \beta^5) = 11(\alpha^3 + \beta^3)$$

$$\Rightarrow \quad \frac{\alpha^5 + \beta^5}{\alpha^3 + \beta^3} = \frac{11}{3}$$

$$\Rightarrow \quad \frac{(\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)}{\alpha^3 + \beta^3} = \frac{11}{3}$$

$$\Rightarrow \quad (\alpha^2 + \beta^2) - \frac{\alpha^2\beta^2(\alpha + \beta)}{(\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)} = \frac{11}{3}$$

$$\Rightarrow \quad 5 - \frac{\alpha^2\beta^2}{5 - \alpha\beta} = \frac{11}{3} \qquad [\because \alpha^2 + \beta^2 = 5]$$

$$\Rightarrow \quad \frac{\alpha^2\beta^2}{5 - \alpha\beta} = 5 - \frac{11}{3}$$

$$\Rightarrow \quad \frac{\alpha^2\beta^2}{5 - \alpha\beta} = \frac{4}{3}$$

$\Rightarrow$ $3(\alpha\beta)^2 + 4(\alpha\beta) - 20 = 0$

$\Rightarrow$ $3(\alpha\beta)^2 + 10(\alpha\beta) - 6(\alpha\beta) - 20 = 0$

$\Rightarrow$ $(\alpha\beta - 2)(3\alpha\beta + 10) = 0$

$$\Rightarrow \quad \alpha\beta = 2 \text{ or, } \alpha\beta = -\frac{10}{3}.$$

If $\alpha\beta = 2$, then

$\alpha^2 + \beta^2 = 5$

$\Rightarrow$ $(\alpha + \beta)^2 - 2\alpha\beta = 5$

$\Rightarrow$ $(\alpha + \beta)^2 - 4 = 5$

$\Rightarrow$ $(\alpha + \beta)^2 = 9$

$\Rightarrow$ $\alpha + \beta = \pm 3$

So, the quadratic equation is

$$x^2 - x(\alpha + \beta) + \alpha\beta = 0 \text{ or, } x^2 \pm 3x + 2 = 0$$

If $\alpha\beta = -\frac{10}{3}$, then

$$\alpha^2 + \beta^2 = 5$$

$\Rightarrow$ $(\alpha + \beta)^2 - 2\alpha\beta = 5$

$\Rightarrow$ $(\alpha + \beta)^2 + \frac{20}{3} = 5$

$\Rightarrow$ $(\alpha + \beta)^2 = -\frac{5}{3}$, which in not possible as $\alpha, \beta \in R$

Hence, the required quadratic equations are $x^2 \pm 3x + 2 = 0$.