Problem Based on Nature of Roots

VERY SHORT ANSWER TYPE QUESTIONS :

VSA.1 Find the nature of the roots of the following equations without actually solving them :

(a) $4x^2 - 20x + 25 = 0$ (b) $15x^2 - x - 2 = 0$

(c) $2x^2 - 8x + 3 = 0$ (d) $7x^2 + 2x + 11 = 0$

Sol.

(a) We have,

$$4x^2 - 20x + 25 = 0$$

Here, $a = 4, b = -20, c = 25$

$\therefore$ $D = b^2 - 4ac$

$= (-20)^2 - 4(4)(25)$

$= 400 - 400 = 0$

Hence, roots are real and equal.

(b) We have,

$$15x^2 - x - 2 = 0$$

Here, $a = 15, b = -1, c = -2$

$\therefore$ $D = b^2 - 4ac$

$= (-1)^2 - 4(15)(-2)$

$= 1 + 120 = 121 > 0$

Which is greater than zero and is also perfect square $\{(11)^2 = 121\}$.

Hence, roots are rational numbers and are unequal.

(c) $2x^2 - 8x + 3 = 0$

Here, $a = 2, b = -8, c = 3$

$\therefore$ $D = b^2 - 4ac$

$= (-8)^2 - 4(2)(3)$

$= 64 - 24 = 40 > 0$

Which is greater than zero, but it is not a perfect square.

Hence, roots are irrational and unequal.

(d) $7x^2 + 2x + 11 = 0$

Here, $a = 7, b = 2, c = 11$

$\therefore$ $D = b^2 - 4ac$

$= (2)^2 - 4(7)(11)$

$= 4 - 308 = -304 < 0.$

Hence, roots are complex conjugate is pair.

VSA.2 Discuss the nature of the roots of the equations :

(a) $\sqrt{3}x^2 - 2x - \sqrt{3} = 0$ (b) $x^2 - 2\sqrt{3}x - 1 = 0$

(c) $x^2 - (p + 1)x + p = 0;\ P \in Q.$

Sol.

(a) Here, disc = $2^2 - 4\sqrt{3}(-\sqrt{3}) = 16$ is a square of a rational number, but coefficients are not rational so roots are real and unequal.

(b) Here, disc = $(-2\sqrt{3})^2 - 4 \times (1)(-1) = 16$, which is square of rational number and the coefficients are real.

Hence the roots are real and unequal.

(c) Here, disc = $-(p + 1)^2 - 4p = (p - 1)^2$ = square of rational number.

Hence roots are real-rational and unequal unless $p = 1$, when $p = 1$, the roots are rational and unequal.

Problem Based on Nature of Roots

VSA.3 Prove that the roots of the equation $2x^2 - x - \frac{3}{2} = 0$ are irrational.

Sol. We have,

$$2x^2 - x - \frac{3}{2} = 0 \qquad ....(1)$$

Comparing (1) with $ax^2 + bx + c = 0$, we get $a = 2$, $b = -1$, $c = -\frac{3}{2}$

$$\therefore \quad D = b^2 - 4ac$$

$$= (-1)^2 - 4(2)\left(-\frac{3}{2}\right)$$

$$= 1 + 12 = 13 > 0$$

Which is greater than zero and is not a perfect square. Therefore, the roots of equation (1) are irrational and unequal.

VSA.4 For what value of m will the equation $x^2 - 2(1 + 3m)x + 7(3 + 2m) = 0$ have equal roots ?

Sol. If the given equation has equal roots, then its discriminant is zero i.e.

$$4(1 + 3m)^2 - 4 \times 7(3 + 2m) = 0$$

$$\Rightarrow \quad 9m^2 - 8m - 20 = 0$$

$$\Rightarrow \quad (m - 2)(9m + 10) = 0$$

$$\Rightarrow \quad m = 2 \text{ or } m = -\frac{10}{9}$$

Hence, the roots will be equal for $m = 2$ or $m = -\frac{10}{9}$.

VSA.5 Show that the roots of $2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$ are imaginary.

Sol. We have,

$$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$$

Comparing (1) with

$Ax^2 + Bx + C = 0$, we get

$A = 2(a^2 + b^2)$, $B = 2(a + b)$ and $C = 1$

$$D = B^2 - 4AC = 4(a + b)^2 - 4 \times 2(a^2 + b^2) \times 1$$

$$= 4[(a + b)^2 - 2(a^2 + b^2)] = 4[a^2 + b^2 + 2ab - 2a^2 - 2b^2]$$

$$= 4[-a^2 - b^2 + 2ab]$$

$$= -4[a^2 + b^2 - 2ab]$$

$$= -4[a - b)^2$$

$$= D < 0$$

Hence, the roots of the given equation are complex conjugate in pair.

Problem Based on Nature of Roots

SHORT ANSWER TYPE QUESTIONS :

SA.1 If the roots of the equation $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$ be equal, show that $\frac{1}{p} + \frac{1}{r} = \frac{2}{q}$.

Sol. The given equation is

$$p(q - r)x^2 + q(r - p)x + r(p - q) = 0 \qquad .....(1)$$

$$D = [q(r - p)]^2 - 4p(q - r)\,r(p - q)$$

$$= q^2(r - p)^2 - 4pr(q - r)\,(p - q)$$

$$= q^2r^2 + p^2q^2 - 2q^2rp - 4p^2qr + 4q^2pr + 4p^2r^2 - 4pqr^2$$

$$= p^2q^2 + q^2r^2 + 4r^2p^2 + 2q^2pr - 4pqr^2 - 4p^2qr$$

$$= (pq + qr - 2rp)^2$$

We know that equation (1) will have equal roots, if $D = 0$

$\Rightarrow \quad (pq + qr - 2rp)^2 = 0$

$\Rightarrow \quad pq + qr - 2rp = 0$

$\Rightarrow \quad pq + qr = 2rp \qquad .....(2)$

Dividing both the side of (2) by pqr, we get

$$\frac{1}{r} + \frac{1}{p} = \frac{2}{q}.$$

SA.2 Prove that the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are equal if and only if $a = b = c$.

Sol. The given equation is $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$

$\Rightarrow \quad (x^2 - ax - bx + ab) + (x^2 - bx - cx + bc) + (x^2 - cx - ax + ca) = 0$

$\Rightarrow \quad 3x^2 - 2(a + b + c)x + (ab + bc + ca) = 0 \qquad ....(1)$

Comparing (1) with $Ax^2 + Bx + C = 0$, we get

$$A = 3,\ B = -2(a + b + c),\ C = (ab + bc + ca)$$

$\therefore \quad D = B^2 - 4AC$

$$= \{-2(a + b + c)\}^2 - 4(3)(ab + bc + ca)$$

$$= 4[(a + b + c)^2 - 3(ab + bc + ca)]$$

$$= 4[a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca]$$

$$= 4[a^2 + b^2 + c^2 - ab - bc - ca]$$

$$= 2[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca]$$

$$= 2[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)$$

$$= 2[(a - b)^2 + (b - c)^2 + (c - a)^2]$$

The roots of (1) will be equal, if and only if

$$D = 0$$

$\Leftrightarrow \quad 2[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$

$\Leftrightarrow \quad (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Leftrightarrow \quad a - b = 0$ and $b - c = 0$ and $c - a = 0$

$\Leftrightarrow \quad a = b$ and $b = c$ and $c = a$

$\Leftrightarrow \quad a = b = c.$

Problem Based on Nature of Roots

SA.3 Find the value of k for which the equation $(2k + 3)x^2 + 2(k + 3)x + (k + 5) = 0$ has equal roots. Also, solve the equation for those values of k.

Sol. The given equation is

$(2k + 3)x^2 + 2(k + 3)x + (k + 5) = 0$ ....(1)

Comparing (1) with $ax^2 + bx + c = 0$, we get

$a = (2k + 3),\ b = 2(k + 3),\ c = k + 5$

$\therefore$ $D = b^2 - 4ac$

$= \{2(k + 3)\}^2 - 4(2k + 3)(k + 5)$

$= 4(k^2 + 6k + 9) - 4(2k^2 + 13k + 15)$

$= -4k^2 - 28k - 24$

Equation (1) will have equal roots if D = 0.

i.e. if $-4(k^2 + 7k + 6) = 0$

$\Rightarrow$ $(k + 1)(k + 6) = 0$

$\Rightarrow$ $k = -1, -6.$

When $k = -1$, equation (1) becomes

$x^2 + 4x + 4 = 0$

$\Rightarrow$ $(x + 2)^2 = 0$

$\Rightarrow$ $x = -2, -2$

When $k = -6$, equation (1) becomes

$-9x^2 - 6x - 1 = 0$

$\Rightarrow$ $9x^2 + 6x + 1 = 0$

$\Rightarrow$ $(3x + 1)^2 = 0$

$\Rightarrow$ $x = -\frac{1}{3}, -\frac{1}{3}$

SA.4 Determine the positive values of k for which the equations :

$x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ will both have real roots.

Sol. We have,

$x^2 + kx + 64 = 0$ ....(1)

This equation will have real roots, if $D \geq 0$

$\Rightarrow$ $k^2 - 4(1)(64) \geq 0$

$\Rightarrow$ $k^2 \geq 256$

$\Rightarrow$ $k \geq 16$ ....(2)

Also, $x^2 - 8x + k = 0$ will have real roots, if $D \geq 0$.

$\therefore$ $(-8)^2 - 4(1)(k) \geq 0$

$\Rightarrow$ $64 - 4k \geq 0$

$\Rightarrow$ $64 \geq 4k$

$\Rightarrow$ $k \leq 16$ ....(3)

From (2) and (3), we get k = 16.

Therefore, both the given equation will have real roots, if k = 16.

Problem Based on Nature of Roots

SA.5 Find he minimum and maximum values of $\frac{x^2+14x+9}{x^2+2x+3}$, for real values of x.

Sol. Let $\frac{x^2+14x+9}{x^2+2x+3} = k$

$x^2 + 14x + 9 = k(x^2 + 2x + 3)$

$\Rightarrow \quad (k - 1)x^2 + 2(k - 7)x + (3k - 9) = 0$

For real value of x, we have, $D \geq 0$

$\Rightarrow \quad D = [4(k - 7)^2 - 12(k - 1)(k - 3)] \geq 0$

$\Rightarrow \quad -8k^2 - 8k + 160 \geq 0$

$\Rightarrow \quad -8(k^2 + k - 20) \geq 0$

$\Rightarrow \quad -8(k + 5)(k - 4) \geq 0$

$\Rightarrow \quad (k + 5)(k - 4) \geq 0$ [Case I]

$$\Rightarrow \quad \left[\begin{array}{l} \text{Either } k + 5 \geq 0 \text{ and } k - 4 \leq 0 \\ \Rightarrow k \geq -5 \text{ and } k \leq 4 \Rightarrow -5 \leq k \leq 4 \end{array}\right.$$

$\Rightarrow \quad -5 \leq k \leq 4$

$$\text{or} \quad \left[\begin{array}{l} \quad k + 5 \leq 0 \text{ and } k - 4 \geq 0 \\ k \leq -5 \text{ and } k \geq 4 \rightarrow \text{Not possible} \end{array}\right.$$

$\therefore$ Minimum value = –5 and maximum value of = 4.

SA.6 If x be real, show that the expression $\frac{x^2-2x+9}{x^2+2x+9}$ lies between $\frac{1}{2}$ and 2.

Sol. Let $\frac{x^2-2x+9}{x^2+2x+9} = k$. Then,

$x^2 - 2x + 9 = k(x^2 + 2x + 9)$

$\Rightarrow \quad (1 - k)x^2 - 2(1 + k)x + 9(1 - k) = 0$

Since, x is real, $D \geq 0$.

$\Rightarrow \quad 4(1 + k)^2 - 4(1 - k) \times 9(1 - k) \geq 0$

$\Rightarrow \quad (1 + 2k + k^2) - 9(1 - 2k + k^2) \geq 0$

$\Rightarrow \quad -8k^2 + 20k - 8 \geq 0$

$\Rightarrow \quad -4(2k^2 - 5k + 2) \geq 0$

$\Rightarrow \quad 2k^2 - 5k + 2 \leq 0$

$\Rightarrow \quad (2k - 1)(k - 2) \leq 0$

Two case arise :

Case I :

$2k - 1 \geq 0$ and $k - 2 \leq 0$,

i.e., $k \geq \frac{1}{2}$ and $k \leq 2$

i.e., $\frac{1}{2} \leq k \leq 2$.

Case II :

$2k - 1 \le 0$ and $k - 2 \ge 0$

i.e., $k \le \frac{1}{2}$ and $k \ge 2$. Which is not possible.

$\therefore$ $\frac{1}{2} \le k \le 2$

$\Rightarrow$ $\frac{1}{2} \le \frac{x^2 - 2x + 9}{x^2 + 2x + 9} \le 2$.

SA.7 If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + bx + c$, where $ac \ne 0$, then show that $P(x)\,Q(x) = 0$ has at least two real roots.

Sol. Let D_1 and D_2 be the discriminants of $P(x) = 0$ and $Q(x) = 0$ respectively, then $D_1 = b^2 - 4ac$ and $D_2 = b^2 + 4ac$.

Since $ac \ne 0$, therefore, either $ac > 0$ or $ac < 0$.

When $ac > 0$, then $b^2 + 4ac > 0$.

But $D_1 = b^2 - 4ac$ may be positive or negative.

Thus roots of $Q(x) = 0$ are real.

When $ac < 0$, then $b^2 - 4ac > 0$.

But $D_2 = b^2 + 4ac$ may be positive or negative.

Thus roots of $P(x) = 0$ are real.

Hence, at least two roots of $P(x) = Q(x) = 0$ are real.

SA.8 For what value of m, will the following equation have real and equal roots

$x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$?

Sol. The given equation is

$x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$

$\therefore$ $\Delta = b^2 - 4ac$

$= [-2(1 + 3m)]^2 - 4(1)\,[7(3 + 2m)]$

$= 4(1 + 6m + 9m^2) - 28(3 + 2m)$

$= 4 + 24m + 36m^2 - 84 - 56m$

$= 36m^2 - 32m - 80$.

For real and equal roots, $\Delta = 0$

$\Rightarrow$ $36m^2 - 32m - 80 = 0$

$\Rightarrow$ $9m^2 - 8m - 20 = 0$

$\Rightarrow$ $9m^2 - 18m - 10m - 20 = 0$

$\Rightarrow$ $9m(m - 2) - 10(m - 2) = 0$

$\Rightarrow$ $(m - 2)(9m + 10) = 0$

$\Rightarrow$ $m = 2, -\frac{10}{9}$

Hence $m = 2, -\frac{10}{9}$.

Problem Based on Nature of Roots

SA.9 The roots r_1 and r_2 of the quadratic equation : $5x^2 - px + 1 = 0$ are real and distinct. Find all possible values of p.

Sol. The given equation is $5x^2 - px + 1 = 0$

Its roots are r_1 and r_2, which are real and distinct.

$\therefore$ Discriminant Δ $(= b^2 - 4ac) > 0$

$\Rightarrow (-p)^2 - 4(5)(1) > 0$

$\Rightarrow p^2 - 20 > 0$

$\Rightarrow p^2 > 20$

$\Rightarrow |p| > \sqrt{20}$

$\Rightarrow p < -\sqrt{20}$ or $p > \sqrt{20}$

$\Rightarrow p \in (-\infty, -\sqrt{20})$ or $p \in (\sqrt{20}, \infty)$.

Hence, $p \in (-\infty, -\sqrt{20}) \cup (\sqrt{20}, \infty)$.

SA.10 If $x \in R$, prove that $3x^2 + 7x + 10$ cannot be less than $\frac{71}{12}$.

Sol. Let $3x^2 + 7x + 10 = m$

$\Rightarrow 3x^2 + 7x + (10 - m) = 0$ ....(1)

Which is quadratic in x having real roots, therefore, its discriminant ≥ 0.

$\Rightarrow 7^2 - 4.3.(10 - m) \geq 0$

$\Rightarrow -71 + 12m \geq 0$

$\Rightarrow 12m \geq 71$

$\Rightarrow m \geq \frac{71}{12}$.

This shows that for real values of x, the expression $3x^2 + 7x + 10$ cannot be less than $\frac{71}{12}$.

SA.11 Prove that the roots of the equation $(a^4 + b^4)x^2 + 4abcd\,x + (c^4 + d^4) = 0$ cannot be different, if real.

Sol. The discriminant of the given equation is

$$D = 16a^2b^2c^2d^2 - 4(a^4 + b^4)(c^4 + d^4)$$
$$= -4[(a^4 + b^4)(c^4 + d^4) - 4a^2b^2c^2d^2]$$
$$= -4[a^4c^4 + a^4d^4 + b^4c^4 + b^4d^4 - 4a^2b^2c^2d^2]$$
$$= -4[(a^4c^4 + b^4d^4 - 2a^2b^2c^2d^2) + (a^4d^4 + b^4c^4 - 2a^2b^2c^2d^2)]$$
$$= -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \quad ....(1)$$

Since roots of the given equation are real, therefore $D \geq 0$

$\Rightarrow -4[(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \geq 0$

$\Rightarrow [(a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2] \leq 0$

$\Rightarrow (a^2c^2 - b^2d^2)^2 + (a^2d^2 - b^2c^2)^2 = 0$ ....(2)

[$\because$ Sum of two positive quantities cannot be negative]

From (1) and (2), we get D = 0

Hence, the roots of the given quadratic equation are not different if they are real.

Problem Based on Nature of Roots

SA.12 If the roots of equation $ax^2 + 2bx + c = 0$ are real and distinct, then show that the roots of the equation $(a^2 - ac + 2b^2)\,x^2 + 2b\,(a + c)\,x + (c^2 - ac + 2b^2) = 0$ are imaginary.

Sol. The two equations are

$ax^2 + 2bx + c = 0$ ....(1)

and, $(a^2 - ac + 2b^2)\,x^2 + 2b\,(a + c)\,x + (c^2 - ac + 2b^2) = 0$ ....(2)

Let D_1 and D_2 be the discriminants of (1) and (2) respectively. Then,

$D_1 = 4(b^2 - ac)$ ....(3)

and $D_2 = 4\,b^2\,(a + c)^2 - 4(a^2 - ac + 2b^2)\,(c^2 - ac + 2b^2)$

$= 4\,[b^2\,(a + c)^2 - \{(b^2 - ac) + (a^2 + b^2)\}\,\{(b^2 - ac) + (b^2 + c^2)\}$

$= 4\,[b^2\,(a + c)^2 - \{(b^2 - ac)^2 + (b^2 - ac)\,(a^2 + 2b^2 + c^2) + (a^2 + b^2)\,(b^2 + c^2)\}$

$= 4\,[(b^2a^2 + b^2c^2 + 2ab^2c) - (b^2 - ac)^2 - (b^2 - ac)\,(a^2 + 2b^2 + c^2)$

$- (a^2b^2 + b^4 + a^2c^2 + b^2c^2)]$

$= 4\,[(2ab^2c - b^4 - a^2c^2) - (b^2 - ac)^2 - (b^2 - ac)\,(a^2 + 2b^2 + c^2)]$

$= 4\,[-(b^2 - ac)^2 - (b^2 - ac)^2 - (b^2 - ac)\,(a^2 + 2b^2 + c^2)]$

$= 4\,[-2\,(b^2 - ac)^2 - (b^2 - ac)\,(a^2 + 2b^2 + c^2)]$

$=- 4\,(b^2 - ac)\,[2\,(b^2 - ac) + (a^2 + 2b^2 + c^2)]$

$= -4\,(b^2 - ac)\,[4\,b^2 + a^2 + c^2 - 2ac]$

$= -4\,(b^2 - ac)\,[(a - c)^2 + (2b)^2]$

$= -D_1\,[(a - c)^2 + (2b)^2]$ ....(4)

Now, roots of (1) are real and distinct

$\Rightarrow$ $D_1 > 0$

$\Rightarrow$ $D_2 < 0$

$\Rightarrow$ roots of (2) are imaginary.

SA.13 Discuss the nature of roots :

(a) $x^2 - 2\left(m + \dfrac{1}{m^2}\right)x + 3;\ m \in R,\ m \neq 0$

(b) $a(b - c)\,x^2 + b(c - a)x + c(a - b) = 0;\ a,\ b \in Q.$

Sol.

(a) Here, disc $= \left\{-2\left(m + \dfrac{1}{m^2}\right)\right\}^2 - 4 \times 1 \times 3$

$= 4\left(m^2 + \dfrac{1}{m^2} + 2 - 3\right)$

$= 4\left(m^2 + \dfrac{1}{m^2} - 1\right)$

$= 4\left(m^2 + \dfrac{1}{m^2} - 2 + 1\right)$

$= 4\left\{\left(m + \dfrac{1}{m}\right)^2 + 1\right\} \geq 4$

as square of real number cannot be negative.

$\therefore$ Roots are real and unequal.

(b) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0;\ a, b \in Q$.

Given equation is of the form

$$Ax^2 + Bx + C = 0$$

Where $A = a(b - c),\ B = b(c - a),\ C = c(a - b)$, so that

$$A + B + C = 0$$

$\Rightarrow$ $B = -A - C$

Here,
$$\begin{aligned}\text{disc} &= B^2 - 4AC\\ &= (-A - C)^2 - 4AC\\ &= A^2 + C^2 - 2AC - 4AC\\ &= A^2 + C^2 - 2AC\\ &= (A - C)^2\end{aligned}$$

Roots are real rational and unequal unless $ab + bc = 2ac$; when $ab + bc = 2ac$, the roots are real-rational and equal.

SA.14 Determine the discriminant of the equations :

$$ax^2 + (a - b)x = (b + c - a) \text{ and } cx^2 + a(2x + 1) = 0.$$

Hence, show that the roots of the former are real and distinct provided those of the latter are imaginary.

Sol. The given equations are

$$ax^2 + (a - b)x\ (-(b + c - a)) = 0 \qquad \text{..........(1)}$$

and $$cx^2 + a(2x + 1) = 0 \qquad \text{..........(2)}$$

disc. of (1) =

$$\begin{aligned}\Delta_1 &= (a - b)^2 - 4(a)(-b + c - a)\\ &= a^2 + b^2 - 2ab + 4ab + 4ac - 4a^2\\ &= a^2 + b^2 + 2ab + 4ac - 4a^2\\ &= (a + b)^2 + 4a(c - a).\end{aligned}$$

$\therefore$ $$\Delta_1 = (a + b)^2 + 4a(c - a) \qquad \text{..........(3)}$$

disc. of (2) =

$$\begin{aligned}\Delta_2 &= (2a)^2 - 4(a)(c)\\ &= 4a^2 - 4ac\\ &= -4a(c - a)\end{aligned}$$

$\therefore$ $$\Delta_2 = -4a(c - a) \qquad \text{..........(4)}$$

The roots of (2) are given to be imaginary.

$\therefore$ $\Delta_2 < 0$

$\therefore$ $-4a(c - a) < 0$

$\Rightarrow$ $4a(c - a) > 0$

Now, $\Delta_1 = (a + b)^2 + 4a(c - a) > 0$ $\quad [\because (a + b)^2 \geq 0 \text{ and } 4a(c - a) > 0]$

$\therefore$ The roots of (1) are real and distinct.